Notes on Feynman Diagrams for Scattering Problems in One Dimension

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Formulation	Discoverer	Computation of Amplitudes	Best suited for	Co
Wave Mechanics	Schrödinger	Solve Wave Equation	One-dimensional Potentials of Arbitrary Form	8.0
Matrix Mechanics	Heisenberg	Operator Algebra	SHO, Angular Momentum, Spin	8.0
Sum over Histories	Feynman	Feynman Diagrams	Field Theory, Many-body physics	8.3

Table 1: Independent, but equivalent, Formulations of Quantum Mechanics

1 Introduction

There are three *independent* formulations of quantum mechanics. Although all three are known to be equivalent, they are quite different in terms of the physics which they emphasise, and a given problem is usually most directly attacked within one but not the others. The unifying features shared by all of these formulations deal with the probabilistic nature of fundamental events which we have discussed. All three formalations of Quantum mechanics include the following three features a) events are viewed as probabilistic, b) the probability for the occurence of an event is given by the complex magnitude squared of the *quantum amplitude* for the event, and c) a perscription for computing the quantum amplitude associated with an event. We know that all three formulations are equivalent because it may be shown that they give the sample quantum amplitudes. Table 1 summaries thes three formulations.

The first formulation is the one in which we have worked primarily in 8.04, Wave Mechanics. This formulation, where amplitudes are computed by solving a wave equation, is due to Schrödinger. This formulation is most convenient when working with potentials of arbitrary form in one dimension. The second formulation is Matrix Mechanics, and is due to Heisenberg. We have had some exposure to this formulation where quantum amplitudes are are computed using operator algebra. This formulation is most convenient when working with problems with special symmetries such as the Simple Harmonic Oscillator, problems involving angular momentum and problems involving the spin of particles. In 8.05 and 8.059, you will be exposed heavily to this formulation which is usually carried out using Dirac's notation. The third and final formulation is due to Feynman. In this formulation, one computes ampltidudes for events by summing over all ways in which the event could happed. Practically, this is all done by working with Feynman diagrams. This formulation is extremely powerful and is mostly used in quantum field theory and in the study of many-particle systems such as those found in condensed matter physics. You will use these methods in 8.323, 8.512 and 8.334.

2 The Sum Over Histories

The Sum Over Histories formulation of Quantum Mechanics builds two central ideas of quantum mechanics, superposition and the classical limit into the computation of quantum amplitudes. These two ideas are built directly into the two parts of the the Sum Over Histories Approach:

1. *The quantum amplitude associated with an event is the* sum *of the amplitudes assocatiated with each history leading to the event.*

2. *The quantum amplitude associated with a given history is the product of the amplitudes associated with each fundamental event in the history.*

Item (1) builds superposition, and thus the possibility of quantum intereference, directly into the formulation by specifying. In this context, a "history" is the sequence of events leading to the event in question. This part of the formulation allows for phenomena such as the interference in the two-slit experiment.

Item (2) builds in the classical limit by allowing recovery of the classical notion phyiscs that the probability of an *independent* sequence of events is the product of the probabilities for each event in the sequence, in those cases where quantum intereference effects are unimportant.

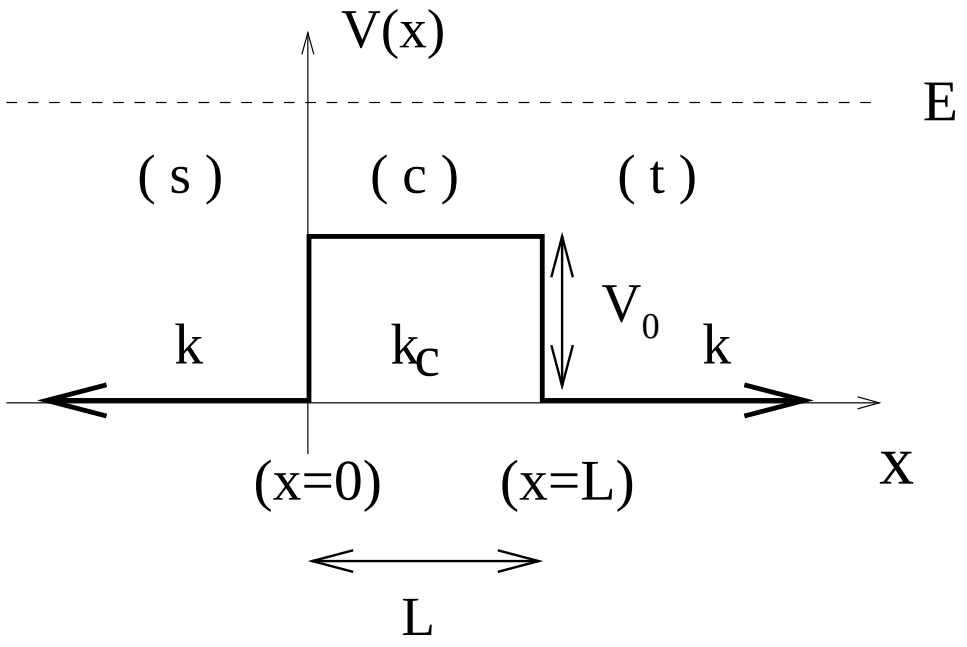


Figure 1: A potential barrier

The appeal of this approach is that it gives a direct perscription for the calcualtion of any quantum amplitude. Moreover, this formulation makes very clear the interface between the physical world and our mathematical description of it. We must use our physical knowledge to determine which are the fundamental events and precisely what amplitudes are to be associated with them. Doing this is analogous to the proceedure we went through to produce the the forms for our operators and the Schrödinger Equations. Another great appeal to the sum over histories approach is that often we are able to intuit the nature and amplitudes of the fundamental events. We will see this procedure in actin in the next section.

3 Example: Scattering from a Step

3.1 Feynman Diagrams

Let us apply the Sum Over Histories approach to compute the transmission probability from the source region s to the transmission region t for the potential barrier occupying the collision region c in Figure 1.

According to (1) from Section 2, we must first lay out all histories, sequences of fundamental events, which could lead to the particle being transmitted from s to t.

The first possibility is that the particle travels direcely across first being transmitted (t) across the upward step at $x = 0$, propagating (p) across to the downward step at $x = L$, from where it is finally transmitted (t'). There are more complicated possibilities, however. The particle could, instead, be reflected (r') from the downward step at $x = L$, propagate back across (p') to the step at $x = 0$, relfect from there (r), propagate (p) to the step at $x = L$ and then finally be transmitted (t). There, in fact, could be any number of such reflections back and forth inside the region of the barrier. Rather than spell these possible "histories" out in words, we represent them as a seqeunce of diagrams as in Figure 2. The two possibililites which we did discuss are labeled 1 and 2 in the figure, respectively. In thus laying out the histories, we have also identified the fundamental events,

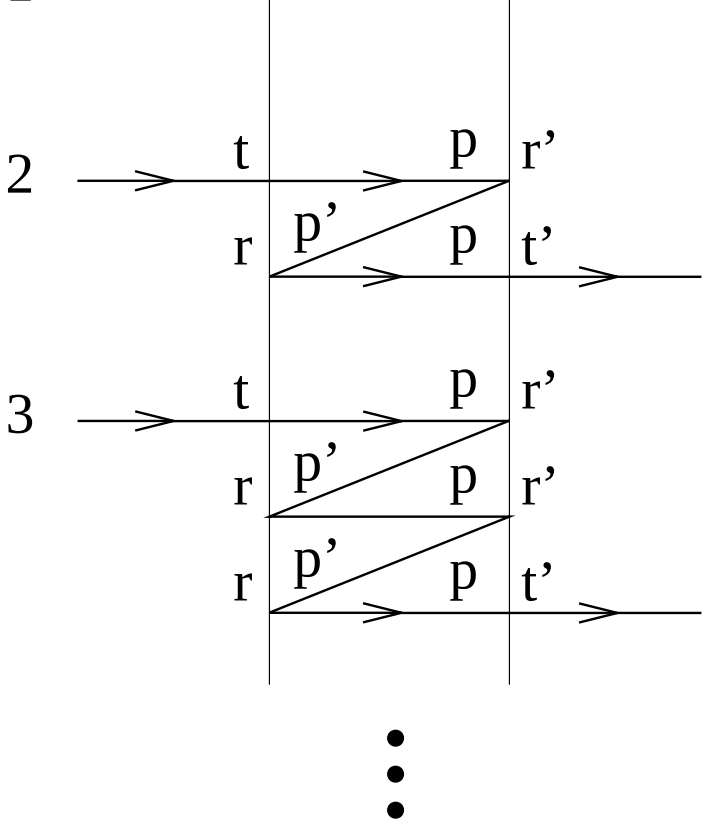


Figure 2: List of Histories Leading to a Transmission Event

transmission through the upward step, t; relfection from the upward step, r, transmission through the downward step, t'; reflection from the downward step, r'; propagation from $x = 0$ to $x = L$, p; and propagation from $x = L$ to $x = 0$.

This set of pictures in Figure 2 are the *Feynman diagrams* for scattering from the step.

3.2 Feynman Rules

Our next task is to assign quantum amplitudes to each fundamental event. What exactly these amplitudes is usually clear from physical reasoninig. These values are the *Feynman rules* for interpreting our Feynman diagrams and translating the diagrams into actual quantum amplitudes.

For our choice of t, the transmission amplitude across the upward step at $x = 0$, we must be able to compute the correct transmission probabililty for the simple step pictured in Figure 3. In this case there is only one history containing the one event t whose amplitude we wish to determine. To produce the correct transmisition probabilities in this simple problem, we must choose the value for t so that $|t|^2 = P_t$ where P_t is the transmission probability which we computed for the step in the "Notes on Scattering Theory.", $|t|^2 = (k_c/k)|2k/(k + k_c)|^2$, where k is the wave vector from the incident region, and k_c is the wave vector in the transmitted region of the upward step. This expression, however, determines only the amplitude of t and not its phase. Knowing physically that the amplitude t describes the effects of transmitting a wave across the step and knowning that the phase picked up by the transmitted wave in the scattering problem was given by $2k_s/(k_s + k_c)$, it is reasonable to set the quantum amplitude associated with the transmission to $t = \sqrt{k_c/k}\,(2k_s/(k_s + k_c))$. This value is recorded in Table 2. Later, we shall verify that this is indeed the correct choice. Note that sometimes the "kinematical" factor $\sqrt{k_c/k}$ is dropped because, as we shall see, it is always cancelled by a corresponding factor in t'.

We may also use our step results to determine, r' and t' the amplitudes for reflecting from or transmitting through the downward step at $x = L$ as well. Again we borrow over our previous results for scattering from steps, noting that the location of the absolute zero of potential is phyiscally irrelvant so that that we may use our previous results taking care to note that the wave vector

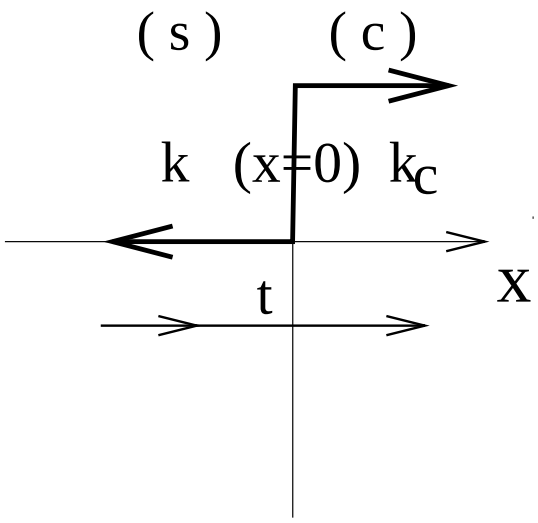


Figure 3: Entire Sum Over Histories for a Single Up-Step Used to Surmize the Value for the amplitude t.

before reaching the step is now k_c and that the wave vector after crossing the step is now k. Also, the fact that the step is centered at $x = L$ instead of $x = 0$ is of no consequence. Physically, the point which we call the origin along the x-axis should not affect the proper values for the quantum amplitudes. Mathematically, if we just center our general solutions to the TISE at the point $x = L$, $(\exp(ik(x - L))$, for instance), we will get the same prefactors on the various waves. The values for the scattering amplitudes of the downward step at $x = L$ are also recorded in Table 2. The factor $\sqrt{k/k_c}$ in t' comes from the kinematical k-dependence in the expression for current density of a beam. This kinetmatical factor will cancel the corresponding factor in t because each diagram contains exactly one factor of t and one factor of t'.

The final scattering factor, r, comes from reflection from the upward step at $x = 0$ but now as we approach this step from the *right*. Physically, this represents the same event as does r', but coming from a different direction. Mathematically, this amplitide is determined by solveing the scattering problem from a step but with *right-incident* boundary conditions. However, that problem is precisely the same problem as scattering from the downward step but with the positive direction along the x-axis reversed. The expression for r is then just the same expression for r' but with the replacements $k \leftarrow -k$ and $k_c \leftarrow -k_c$, which, as expected, gives the same amplitde as for r. The value for r is recorded in Table 2.

The final two factors are p and p' describing the amplitude for propagating across the free space between $x = 0$ and $x = L$. To produce the correct result for the simple problem of propagation in free space, we know that we must have $|p|^2 = |p'|^2 = 1$. To determine the phases of these factors, we use the fact that the de Broglie hypothesis specifies a characteristic wavelength $2\pi/k_c$ for propagation of a particle travelling with momentum $\hbar k_c$. The net quantum phase accumulated as the particle propagates from $x = 0$ to $x = L$ should then be $k_c L$ and so we expect $p = \exp(ik_c L)$. For the reverse process p', we would accumulate a phase $\exp(-ik_c L)$ moving from $x = 0$ to $x = L$. However, in the process described by p', the particle moved from $x = L$ to $x = 0$. With the direction reversed, the sense of the accumulation of phase is reversed and we find $p' = \exp(-ik_c(-L)) = \exp(ik_c L) = p$.

Process	Equivalent Amplitude	Amplitude
t		$\sqrt{\frac{k_c}{k}}\frac{2k}{k+k_c}$
r		$\frac{k_c-k}{k+k_c}$
t'		$\sqrt{\frac{k}{k_c}}\frac{2k_c}{k+k_c}$
r'	r	$\frac{k_c-k}{k+k_c}$
p		e^{ik_cL}
p'		e^{-ik_cL}

Table 2: Feyman Rules for Scattering from a Potential Barrier

This completes the origin of the values recorded in Table 2.

Although we allowed for the possibility that the quantum ampltidues p and p' or r and r' were different, we found that in this case, they are the same. Knowing that these factors are the same, in what follows, we shall ignore these two distinctions. Bear in mind, however, that the amplitudes t and t' are different. This situation is also summarized in Table 2.

3.3 Feynman Sum

The amplitude for each history $1, 2, 3, \ldots$ in Figure 2, is just the product of the ampltidues for the events indicated in the figure, where the values for those amplitudes are given in Table 2. The total ampltitude for transmission T is the sum of the ampltides for all of these diagrams, and is therefore,

$$T = t \cdot pt' + t \cdot (pr'p'r) \cdot pt' + t \cdot (pr'p'r)(pr'p'r) \cdot pt' + \ldots \tag{1}$$

We have written the sum to indicate the general features of each of the histories, namely that they all involve transmission into the barrier region t, some number (possibly zero) of repetitions of the basic ricochet sequence $pr'p'r$, and then a final propagation across the and transmission out of the barrier pt'. Noticing these physical features of the histories gives insight into how to perform the summation mathematically. After factoring the common parts from the histories, we are left with an easily summed standard geometric series,

$$T = t(1 + (pr'p'r) + (pr'p'r)^2 + (pr'p'r)^3 + \ldots)pt' \tag{2}$$

$$= t \cdot \frac{1}{1 - (pr'p'r)} \cdot pt' \tag{3}$$

$$= \frac{tpt'}{1 - (pr'p'r)}. \tag{4}$$

This factoring into common end-point events and a basic repreating seqeunce is a common feauture of most Feynman sums, even in far more complicated theories. Such a basic unit, for instance, appears in field theory and represents the self-energy of the electron and eventuallt leads to the theory of renormalization of the electron mass.

Having performed the sum, we now insert the values for the various amplitudes from Table 2 appropriate for the two steps making up the barrier to find the total quantum ampltitude for transmission,

$$T = \frac{\sqrt{\frac{k_c}{k}}\frac{2k}{k+k_c}e^{ik_cL}\sqrt{\frac{k}{k_c}}\frac{2k_c}{k+k_c}}{1 - e^{ik_cL}\frac{k_c-k}{k+k_c}e^{ik_cL}\frac{k_c-k}{k+k_c}} \tag{5}$$

$$= \frac{4kk_c}{e^{-ik_cL}(k+k_c)^2 - e^{ik_cL}(k_c-k)^2} \tag{6}$$

$$= \frac{4kk_c}{((k+k_c)^2 - (k_c-k)^2)\cos(k_cL) - i\left((k+k_c)^2 + (k_c-k)^2\right)\sin(k_cL)} \tag{7}$$

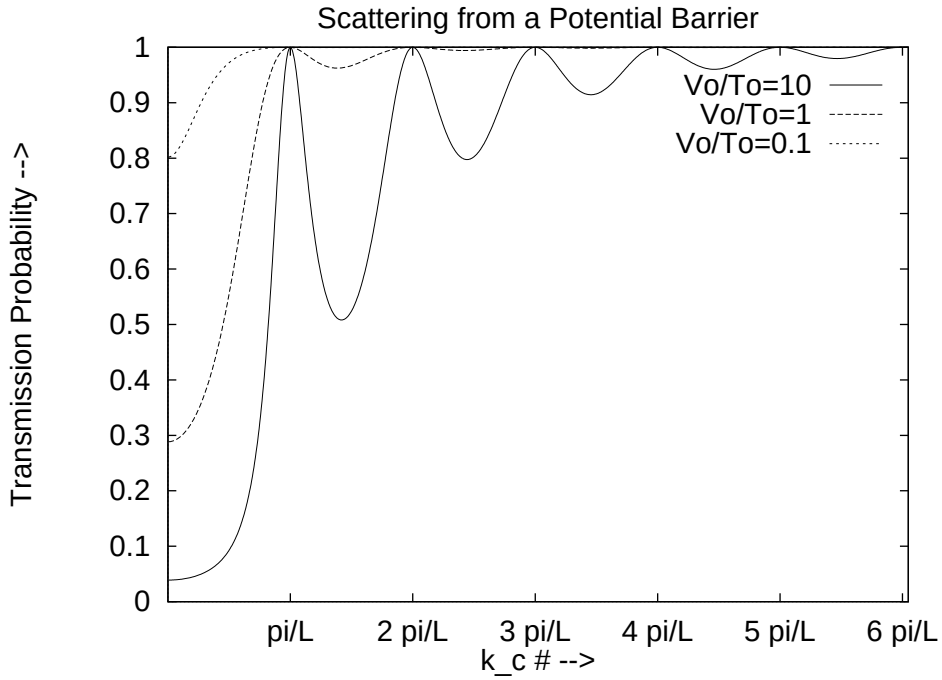


Figure 4: Transmission Probabilities across a square potential barrier

$$= \frac{4kk_c}{(4kk_c)\cos(k_cL) - i\,(2k^2 + 2k_c^2)\sin(k_cL)} \tag{8}$$

$$T = \frac{1}{\cos(k_cL) - \frac{i}{2}\left(\frac{k}{k_c} + \frac{k_c}{k}\right)\sin(k_cL)}, \tag{9}$$

and the corresponding transmission probability,

$$P_T = |T|^2 = \frac{1}{\cos^2(k_cL)} + \frac{1}{4}\left(\frac{k}{k_c} + \frac{k_c}{k}\right)^2 \sin^2(k_cL). \tag{10}$$

One may verify by direct solution, that these are indeed the same results as would be obtained by direct solution of the Time Independent Schrödinger Equation (TISE). We will show how this comes about directly in Section **??**.

In Figure 4, we plot the transmition ampltidude P_T as a function of the momentum k_c of the particle as it crosses the collision region, for various values of the height of the potential step V_o, measured in units of T_o the kinetic energy of a particle bound in an infinite square well potential of length L. Perhaps the most striking result in the figure is that precisely at the values $k_c = n\pi/L$, we find perfect probability for transmission, $P_T = 1$ and all particles are transmitted across the barrier. This phenomena is known as resonant transmisison. We will learn (Section **??**) that this is a very general phenomenon and not peculiar to the sharp edges of our steps. We find also that as expected, when k_c becomes very large, the presense of the barrier becomes less and less significant, so that the transmission probabililty approaches unity at large k_c regardless of whether k_c is at one of the resonances. As we increase the height of the barrier, as we would expect, the transmission probability for low values of k_c decreases rapidly. As a consequence, the resonances become very narrow, as is clearly seen in the figure.

In the next section, we will use the sum over histories approach to show that the resonant phenomena is not unique to the step, but is a general phenomena observed whenever a classically allowed region is sandwitched between two symmetric scatterers.

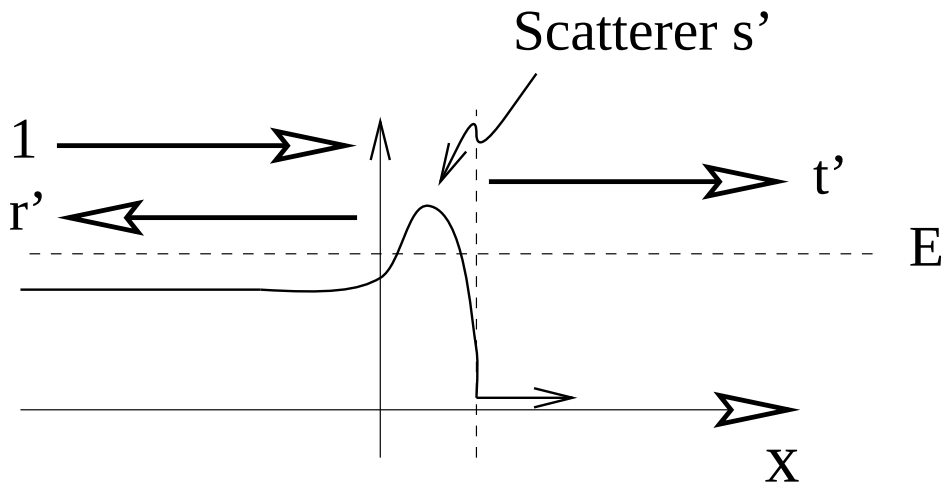


Figure 5: General Resonant Scatterer

4 Resonant Transmission

The resonances evident in Figure 4 represent a very general phenomenon in scattering. They are not particular to the square form of the potential we analysed in the previous section, but have to do soley with two general features of the scattering barrier, 1) two symmetric scatterers opposing each other separated by 2) a classically allowed region in which phase is accumulated.

4.1 General scattering from two barriers

One great advantage in the Feynman approach is that we may easily generalize our results where the basic structure of problems is sharred in the diagrams, although the details of each individiual situation may be burried in the precise form of the quantum ampltidudes making up the Feyman rules.

Figure 5 shows a very general potential satisfying the criteria of two symmetric opposing scatters spanning a classically allowed resonance region. The figure also shows the histories which we would sum in this case. They are the same scenarios which we have already considered. The only differences are in the values of the scattering amplitudes t, p, r', p', r and t'. The values for these are determined using precisely the same arguments which we followed in the previous section. The propagating amplitudes will be $p = p' = \exp(ik_c L)$ as before. The transmission and reflection amplitudes are now determined by solving the appropriate scattering problems for each scatterer separately. The ampltude t', for instance, is computed by solving the scattering problem in Figure 6b) at unit incidence, an incoming beam of unit quantum amplitude. Note that we have included the proper kinematical factors $\sqrt{k/k_c}$ so that, for instance, $|t'|^2$ gives the transmission probability.

Once these amplitudes are determined, the terms in the Feynman sum are precisely the terms we summed before in (1), and so we naturally find the same result (2),

$$T \quad = \quad t \cdot pt' + t \cdot (pr'p'r) \cdot pt' + t \cdot (pr'p'r)(pr'p'r) \cdot pt' + \ldots$$

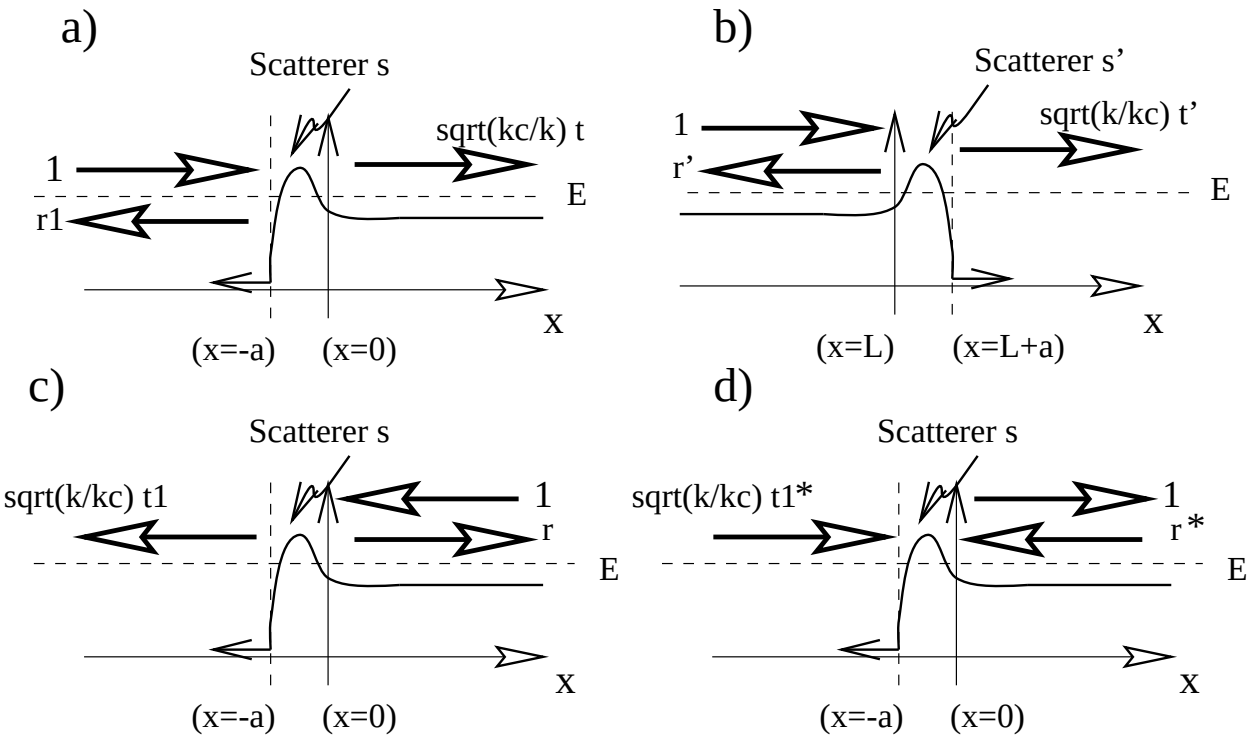


Figure 6: Scattering problem to determine the Feynman rules for barrier scattering: a) Initial transmission into the resonance region t, b) Transmission out of the region or reflection back into it, t' or r', c) Back-reflection into the region r, d) Time-reversal of c)

$$= \frac{tpt'}{1-(pr'p'r)}.$$

This result includes the basic structure of our problem as involving two scatterers separated by a propagating region. To build into the result the fact that the two scatterers are *symmetric*, we must now determine what the symmetry of the scatterers implies about the relationships among t, t', r and r'.

4.2 *Inversion and translation symmetry* constraints on scattering amplitudes

We expect the ampltides t and t' and and r and r' to be related to one another because of the fact that Scatterer s and Scatter s' are mirror images of each other. In fact the potentials for the two scatters $V_s(x)$ and $V_{s'}(x)$, respectively, are related to each other by reflection about the point $L/2$ so that $V_{s'}(x) = V_s(L-x)$. This linear maping may be verified by noting that under this transform the point $-a$ maps to $-a \to L+a$ and the point 0 maps to $0 \to L$, and from Figure 6, these are indeed the corresponding points in the two potentials.

If $\psi_s(x)$ is the scattering solution for $V_s(x)$, by symmetry we expect that $\phi(x) \equiv \psi_s(L-x)$ will be a solution for the scattering problem associated with the potential $V_{s'}(x)$. This is verfied by inserting $\phi(x)$ directly into the TISE,

$$\begin{aligned} -\frac{\hbar^2}{2m}\partial_x^2\phi(x) + V_{s'}(x)\phi(x) &= \\ &= -\frac{\hbar^2}{2m}\partial_x\left(\partial_x\psi_s(L-x)\right) + V_s(L-x)\phi_s(L-x) \\ &= -\frac{\hbar^2}{2m}\partial_x\left(-\psi_s'(L-x)\right) + V_s(L-x)\phi_s(L-x) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{\hbar^2}{2m}\psi_s''(L-x) + V_s(L-x)\phi_s(L-x) \\
&= E\psi_s''(L-x) \quad \text{; from the TISE for state } \psi_x \\
&= E\phi(x) \quad (!)
\end{aligned}
$$

The means that if we start with the $\psi_s(x)$ pictured in 6c),

$$
\psi_s(x) = \begin{cases} \sqrt{k/k_c}t_1 e^{ik(x+a)} & x < -a \\ ?(x) & -a < x < 0 \\ e^{-ik(x+a)} + re^{ik(x+a)} & 0 < x \end{cases} ,
$$

and then form $\phi(x) \equiv \psi(L-x)$, we find a solution for Scatterer s' of the form

$$
\phi(x) = \begin{cases} \sqrt{k/k_c}t_1 e^{ik(L-x+a)} & x < -a \\ ?(L-x) & -a < x < 0 \\ e^{-ik(L-x+a)} + re^{ik(L-x+a)} & 0 < x \end{cases} = \begin{cases} \sqrt{k/k_c}t_1 e^{-ik(x-(L+a))} & L+a < x \\ ?(L-x) & L < x < L+a \\ e^{-ik(x-(L+a))} + re^{-ik(x-(L+a))} & 0 < x \end{cases} .
$$

4.3 *Time reversal symmetry* constraints on scattering amplitudes

The Time Independent Schrödinger Equation, which we solve to determine the scattering amplitudes is a real differential equation,

$$
-\frac{\hbar^2}{2m}\partial_x^2\psi(x) + V(x)\psi(x) = E\psi(x).
$$

The fact that the TISE is a real equation, is a new symmetry which we may exploit. Taking the complex conjugate of both sides of this equation, we get another equation,

$$
-\frac{\hbar^2}{2m}\partial_x^2\psi^*(x) + V(x)\psi^*(x) = E\psi^*(x).
$$

We thus see that if $\psi(x)$ is a solution to the TISE, then so is $\psi^*(x)$. For scattering states, where there are two linearly independent solutions, taking the complex conjugate is a convenient way to produce a physically distinct state.

To see how this works,